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Curvature-dimension condition and heat flow on metric measure spaces

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The *curvature-dimension condition* $\mathsf{CD}(K,N)$, introduced by Sturm and Lott-Villani, is a generalized notion of the combination of a 'lower Ricci curvature bound' (Ric $\geq K$) and an 'upper dimension bound' (dim $\leq N$). Metric measure spaces satisfying CD enjoy many nice properties and well investigated from analytic and geometric points of view. In this note we give a short review on CD and heat flow on metric measure spaces satisfying CD. We refer to surveys [Lo], [Oh3] and the book [Vi2] for more about CD, while this note is also concerned with more recent development.

1 Prehistory

We begin with some precursors of the curvature-dimension condition.

1.1 Wasserstein spaces

We need to review basic notions in *optimal transport theory*, for which we refer to [Vi1] and [Vi2]. Let (X,d) be a complete separable metric space, and denote by $\mathcal{P}(X)$ the space of Borel probability measures on X. We also define $\mathcal{P}^2(X)$ as the subset of $\mathcal{P}(X)$ such that $\mu \in \mathcal{P}^2(X)$ if $\int_X d(x,y)^2\,\mu(dy) < \infty$ for some (and hence all) $x \in X$.

Definition 1.1 (Wasserstein spaces) For $\mu,\nu \in \mathcal{P}^2(X)$, the L^2-*Wasserstein distance* of μ and ν is defined by

$$W_2(\mu,\nu) := \inf_{\pi} \left(\int_{X \times X} d(x,y)^2\,\pi(dxdy) \right)^{1/2},$$

where $\pi \in \mathcal{P}(X \times X)$ runs over all *couplings* of μ and ν, i.e., $\pi(A \times X) = \mu(A)$ and $\pi(X \times A) = \nu(A)$ for every Borel set $A \subset X$. We call $(\mathcal{P}^2(X), W_2)$ the L^2-*Wasserstein space* over X.

In view of optimal transport theory, $d(x,y)^2$ is the *cost* we pay for transporting the unit mass from x to y, $\pi(x,y)$ represents the mass transported from x to y, and $W_2(\mu,\nu)^2$ is the least cost for transporting μ to ν. A *minimal geodesic* with respect to W_2 (i.e., $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}^2(X)$ with $W_2(\mu_s,\mu_t) = |s-t|W_2(\mu_0,\mu_1)$ for all $s,t \in [0,1]$) describes an optimal way of transport.

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If we fix a Borel measure ω on X, then $\mathcal{P}_{\mathrm{ac}}(X,\omega)$ will denote the set of $\mu \in \mathcal{P}(X)$ which is absolutely continuous with respect to ω.

Definition 1.2 (Relative entropy) Define the *relative entropy* $\mathrm{Ent}_\omega(\mu)$ of $\mu \in \mathcal{P}(X)$ with respect to ω by

$$\mathrm{Ent}_\omega(\mu) := \int_{\mathrm{supp}\,\rho} \rho \log \rho \, d\omega,$$

provided that $\mu = \rho\omega \in \mathcal{P}_{\mathrm{ac}}(X,\omega)$ and $\int_{\{\rho>1\}} \rho \log \rho \, d\omega < \infty$. Otherwise, set $\mathrm{Ent}_\omega(\mu) := \infty$.

Observe that, for a Borel set $A \subset X$ with $0 < \omega(A) < \infty$, the uniform distribution $\mu_A := \omega(A)^{-1} \cdot \omega|_A$ on A satisfies $\mathrm{Ent}_\omega(\mu_A) = -\log(\omega(A))$.

1.2 McCann's displacement convexity

Let $\mathbf{L}$ be the Lebesgue measure on $\mathbb{R}^n$. McCann's following pioneering theorem means that $\mathbb{R}^n$ is 'nonnegatively curved'.

Theorem 1.3 (Convexity of $\mathrm{Ent}_{\mathbf{L}}$; [Mc1]) *The relative entropy* $\mathrm{Ent}_{\mathbf{L}}$ *is* convex *on* $(\mathcal{P}^2(\mathbb{R}^n), W_2)$ *in the sense that*

$$\mathrm{Ent}_{\mathbf{L}}(\mu_t) \le (1-t)\,\mathrm{Ent}_{\mathbf{L}}(\mu_0) + t\,\mathrm{Ent}_{\mathbf{L}}(\mu_1) \tag{1.1}$$

for all $t \in [0,1]$ *along any minimal geodesic* $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}^2(\mathbb{R}^n)$ *with respect to* W_2.

We sometimes call (1.1) the *displacement convexity* in order to emphasis the difference from the convexity along the convex combination $t \longmapsto (1-t)\mu_0 + t\mu_1$.

One may understand the validity of the convexity of $\mathrm{Ent}_{\mathbf{L}}$ from the geometric viewpoint as follows. The classical *Brunn-Minkowski inequality* on $\mathbb{R}^n$ asserts that, for any Borel sets $A, B \subset \mathbb{R}^n$ and $t \in [0,1]$,

$$\mathbf{L}\big((1-t)A + tB\big)^{1/n} \ge (1-t)\mathbf{L}(A)^{1/n} + t\mathbf{L}(B)^{1/n}. \tag{1.2}$$

Since the function $s \longmapsto \log(s)$ is increasing and concave, (1.2) immediately implies

$$\mathrm{Ent}_{\mathbf{L}}(\mu_{(1-t)A+tB}) \le (1-t)\,\mathrm{Ent}_{\mathbf{L}}(\mu_A) + t\,\mathrm{Ent}_{\mathbf{L}}(\mu_B).$$

Hence (1.1) can be regarded as a weaker (dimension-free) form of the Brunn-Minkowski inequality.

1.3 Ricci curvature and convexity of relative entropy

Let (M,g) be a complete Riemannian manifold, and denote by ω_g its Riemannian volume measure. We always assume that M is connected and boudaryless.

In their influential paper [OV], Otto and Villani gave a heuristic argument (to be made rigorous in [LV2]) on how a lower Ricci curvature bound implies several functional inequalities via the convexity of the relative entropy Ent_{ω_g}. Their discussion was based on formal calculus in terms of *Otto's Riemannian structure* of $(\mathcal{P}^2(M), W_2)$ ([Ot]).

Then, Cordero-Erausquin et al [CMS] showed up with a rigorous connection between the lower Ricci curvature bound and the convexity of Ent_{ω_g}. We say that $\mathrm{Ric}_g \geq K$ holds for some $K \in \mathbb{R}$ if $\mathrm{Ric}_g(v,v) \geq K|v|^2$ for all $v \in TM$, where Ric_g denotes the Ricci curvature tensor of g.

Theorem 1.4 (Ricci bound implies convexity; [CMS]) *Let (M,g) be compact. If $\mathrm{Ric}_g \geq K$ for some $K \in \mathbb{R}$, then Ent_{ω_g} is K-convex on $(\mathcal{P}(M), W_2)$, i.e.,*

$$\mathrm{Ent}_{\omega_g}(\mu_t) \leq (1-t)\,\mathrm{Ent}_{\omega_g}(\mu_0) + t\,\mathrm{Ent}_{\omega_g}(\mu_1) - \frac{K}{2}(1-t)tW_2(\mu_0,\mu_1)^2 \tag{1.3}$$

holds for all $t \in [0,1]$ along any minimal geodesic $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}(M)$.

We remark that, on a Riemannian manifold, any pair $\mu,\nu \in \mathcal{P}^2_{\mathrm{ac}}(M,\omega_g)$ is connected by a *unique* minimal geodesic (see [Mc2]).

2 Curvature-dimension condition

The converse implication of Theorem 1.4 also holds true.

Theorem 2.1 (Ricci bound is equivalent to convexity; [vRS]) *For $K \in \mathbb{R}$ and a complete Riemannian manifold (M,g), $\mathrm{Ric}_g \geq K$ holds if and only if Ent_{ω_g} is K-convex.*

Note that the K-convexity of Ent_{ω_g} is formulated without using the differentiable structure of M, we need only a distance and a measure. Thus Theorem 2.1 led us to the following notion of 'metric measure spaces with lower Ricci curvature bounds'. Let (X,d,ω) be a complete, separable metric space equipped with a Borel measure ω on X such that $0 < \omega(U) < \infty$ for any nonempty, bounded open set $U \subset X$.

Definition 2.2 (Curvature-dimension condition; [St1], [LV2]) For $K \in \mathbb{R}$, we say that (X,d,ω) satisfies the *curvature-dimension condition* $\mathsf{CD}(K,\infty)$ if any pair $\mu_0,\mu_1 \in \mathcal{P}^2(X)$ is connected by a minimal geodesic $(\mu_t)_{t\in[0,1]} \subset \mathcal{P}^2(X)$ along which (1.3) holds for all $t \in [0,1]$.

Remark 2.3 (a) In general, minimal geodesics in the Wasserstein space are not unique (e.g., over the normed space $(\mathbb{R}^n, |\cdot|_\infty)$). A reason why we impose (1.3) only along *some* minimal geodesic is to make this condition stable under the convergence of underlying spaces (see Theorem 2.4 below).

(b) The word "curvature-dimension condition" comes from Bakry and Émery's celebrated theory on linear semigroups and functional inequalities (see [BE]). We can introduce $\mathsf{CD}(K,N)$ for general $K \in \mathbb{R}$ and $N \in (1,\infty]$ (but the definition is more involved), and a Riemannian manifold (M,g,ω_g) satisfies $\mathsf{CD}(K,N)$ if and only if $\mathrm{Ric}_g \geq K$ and $\dim M \leq N$ ([St2], [LV1]).

(c) If (X,d,ω) satisfies $\mathsf{CD}(K,N)$, then it also satisfies $\mathsf{CD}(K',N')$ for any $K' < K$ and $N' > N$.

There are a number of geometric and analytic applications of $\mathsf{CD}(K,N)$, including (see [St2], [LV1], [LV2] for details):

- *Bishop-Gromov volume comparison theorem* for $N < \infty$;
- *Talagrand, log-Sobolev,* and *global Poincaré inequalities* for $K > 0$ and $N = \infty$;
- *Bonnet-Myers diameter bound* and *Lichnerowicz inequality* for $K > 0$ and $N < \infty$.

Another, geometric motivation behind the study of CD stems from the *Gromov-Fukaya pre-compactness* ([Gr], [Fu]), which asserts that a sequence $\{(M_i, g_i)\}_{i\in\mathbb{N}}$ of complete Riemannian manifolds with uniform bounds $\mathrm{Ric}_{g_i} \ge K$ and $\dim M_i \le N < \infty$ contains a subsequence convergent in the sense of the (pointed) *measured Gromov-Hausdorff convergence.* Such a limit space is not necessarily a Riemannian manifold any more, but still possesses nice properties as was established in Cheeger and Colding's series of works ([CC]). By the following stability theorem, limit spaces certainly satisfy $\mathsf{CD}(K, N)$.

Theorem 2.4 (Stability; [St1], [LV2]) *Suppose that a sequence of metric measure spaces $\{(X_i, d_i, \omega_i)\}_{i\in\mathbb{N}}$ converges to a metric measure space (X, d, ω) in the sense of the (pointed) measured Gromov-Hausdorff convergence. If (X_i, d_i, ω_i) satisfies $\mathsf{CD}(K, N)$ for some $K \in \mathbb{R}$, $N \in (1, \infty]$ and all i, then (X, d, ω) also satisfies $\mathsf{CD}(K, N)$.*

However, it turned out that CD covers a much wider class than the closure of Riemannian manifolds. A *Finsler manifold* (M, F) is a generalization of a Riemannian manifold such that $F : TM \longrightarrow [0, \infty)$ gives a (sufficiently smooth, convex) norm on each tangent space T_xM (see [BCS] for more details).

Theorem 2.5 (Finsler case; [Oh2]) *Let (M, F) be a forward complete Finsler manifold equipped with a positive C^∞-measure ω on M. Then $\mathrm{Ric}_N \ge K$ holds if and only if (M, d_F, ω) satisfies $\mathsf{CD}(K, N)$, where Ric_N is the* weighted Ricci curvature *with respect to ω, and d_F is the distance function induced from F.*

In particular, any normed space $(\mathbb{R}^n, |\cdot|, \mathbf{L})$ satisfies $\mathsf{CD}(0, n)$, whereas normed spaces (other than inner product spaces) can not appear as the limit of Riemannian manifolds. On the one hand, this is a good news since CD is available in a wide class of spaces. On the other hand, one can expect a stronger condition than CD that rules out normed spaces and has stronger consequences. The key ingredient to answer this question is the behavior of heat flow.

3 Heat flow as gradient flow

We can introduce *heat flow* as *gradient flow* in the following two ways:

(I) the gradient flow of the energy $\mathcal{E}$ in the L^2-space;

(II) the gradient flow of the relative entropy in the L^2-Wasserstein space.

The first approach (I) is classical. On a general metric measure space (X, d, ω), $\mathcal{E}$ is introduced as the *Cheeger energy*:

$$\mathcal{E}(u) := \frac{1}{2}\int_X |\nabla u|^2 \, d\omega, \qquad |\nabla u|(x) := \limsup_{y\to x} \frac{|u(x) - u(y)|}{d(x, y)},$$

for a locally Lipschitz function $u : X \longrightarrow \mathbb{R}$ (see [Ch], [AGS1] for the precise definition).

The second, much newer approach (II) was initiated by Jordan, Kinderlehrer and Otto's seminal work ([JKO]). The identification (I)=(II) was then extended to Riemannian manifolds ([Oh1], [Sa], [Er]), Finsler manifolds ([OS1]), Alexandrov spaces ([GKO]), and finally to general metric measure spaces satisfying CD ([AGS1]).

On a non-Riemannian Finsler manifold, however, heat flow is nonlinear. The nonlinearity causes several difficulties in applications. Then, it would be worthwhile to consider spaces satisfying CD such that heat flow is linear, that turned out a nice condition.

4 Riemannian curvature-dimension condition

A metric measure space (X, d, ω) is said to satisfy *strong* $\mathsf{CD}(K, \infty)$ if (1.3) holds along *every* minimal geodesic $(\mu_t)_{t\in[0,1]}$ in $(\mathcal{P}^2(X), W_2)$. For instance, $(\mathbb{R}^n, |\cdot|_\infty, \mathbf{L})$ satisfies $\mathsf{CD}(0, n)$ but does not satisfy strong $\mathsf{CD}(0, \infty)$.

Definition 4.1 (Riemannian curvature-dimension condition; [AGS2]) We say that a metric measure space (X, d, ω) satisfies the *Riemannian curvature-dimension condition* $\mathsf{RCD}(K, \infty)$ if it satisfies strong $\mathsf{CD}(K, \infty)$ and the heat flow on it is linear.

Similarly to CD, $\mathsf{RCD}(K, \infty)$ is preserved under the (pointed) measured Gromov-Hausdorff convergence ([AGS2]). Since Riemannian manifolds (M, g, ω_g) with $\mathrm{Ric}_g \ge K$ satisfy $\mathsf{RCD}(K, \infty)$, their limit spaces also satisfy $\mathsf{RCD}(K, \infty)$. The following characterizations of $\mathsf{RCD}(K, \infty)$ are useful and inspiring.

Theorem 4.2 (Equivalent conditions to RCD; [AGS2]) *The condition* $\mathsf{RCD}(K, \infty)$ *is equivalent to*:

(i) *strong* $\mathsf{CD}(K, \infty)$ *and that* $\mathcal{E}$ *is quadratic (so that* $\mathcal{E}$ *induces a Dirichlet form)*;

(ii) *the* evolution variational inequality:

$$\frac{d}{dt}\left[\frac{W_2(\mu_t, \nu)^2}{2}\right] + \frac{K}{2} W_2(\mu_t, \nu)^2 + \mathrm{Ent}_\omega(\mu_t) \le \mathrm{Ent}_\omega(\nu)$$

for all $(\mu_t)_{t>0} \subset \mathcal{P}^2(X)$ *obeying heat flow, all* $\nu \in \mathcal{P}^2(X)$*, and for a.e.* $t > 0$.

Roughly speaking, the evolution variational inequality is derived by estimating the first variation $\frac{d}{dt}[W_2(\mu_t, \nu)^2/2]$ by the K-convexity of Ent_ω along a minimal geodesic between μ_t and ν, for which we essentially need the 'Riemannian structure' (i.e., angles). In a similar manner, we can control the distance between two curves obeying heat flow.

Theorem 4.3 (K-contraction property) *Let us assume* $\mathsf{RCD}(K, \infty)$. *Then, for any curves* $(\mu_t)_{t\ge0}, (\nu_t)_{t\ge0} \subset \mathcal{P}^2(X)$ *along heat flow, we have*

$$W_2(\mu_t, \nu_t) \le e^{-Kt} W_2(\mu_0, \nu_0) \qquad \forall t > 0.$$

This is a standard consequence on the gradient flow of a K-convex function, while we need the 'Riemannian structure' for the same reason as the evolution variational inequality. Thanks to Kuwada's duality ([Ku]) on linear semigroups, the K-contraction property implies the following.

Theorem 4.4 (Bakry-Émery type gradient estimate) *Suppose* $\mathsf{RCD}(K,\infty)$ *and let* $(\rho_t)_{t\ge 0}$ *be along heat flow with* $\rho_t\omega \in \mathcal{P}^2(X)$. *Then we have*

$$|\nabla \rho_t|(x)^2 \le e^{-2Kt} P_t(|\nabla \rho_0|^2)(x) \qquad \forall x \in X,\ \forall t > 0,$$

where P_t *is the heat semigroup (i.e.,* $P_t(|\nabla \rho_0|^2)$ *is the heat flow starting from* $|\nabla \rho_0|^2$*).*

Furthermore, we obtain the (dimension-free) *Bochner inequality*

$$\frac{1}{2}\Delta(|\nabla u|^2) - D(\Delta u)(\nabla u) \ge K|\nabla u|^2$$

in a certain weak sense ([GKO], [AGS2]).

5 Further problems

5.1 CD vs Finsler

Recall that a Finsler manifold satisfies CD but does not satisfy RCD unless it is a Riemannian manifold. Therefore Finsler manifolds are reasonable model spaces satisfying CD. The following theorem reveals the difference between Riemannian and Finsler manifolds (in other words, the difference between RCD- and CD-spaces).

Theorem 5.1 (Non-contraction of heat flow; [OS2]) *The heat flow of a normed space* $(\mathbb{R}^n, |\cdot|, \mathrm{L})$ *is not* K*-contractive for any* $K \in \mathbb{R}$ *unless the norm* $|\cdot|$ *comes from an inner product.*

We can nevertheless show the *Bochner-Weitzenböck formula* on general Finsler manifolds, and as applications the (modified versions of) Bakry-Émery type and *Li-Yau type gradient estimates* as well as the *Harnack inequality* ([OS3]). Furthermore, the *Cheeger-Gromoll type (homeomorphic) splitting theorem* was generalized in [Oh4].

For general CD-spaces, Gigli recently showed the Laplacian comparison theorem ([Gi]). The Bochner-Weitzenböck formula and the gradient estimates are not known on general CD-spaces.

5.2 $\mathsf{RCD}(K,N)$ for $N<\infty$?

It is unclear how to define $\mathsf{RCD}(K,N)$ for $N<\infty$ (especially with $K \neq 0$). $\mathsf{RCD}(K,\infty)$ is naturally related to the behavior of heat flow via the relative entropy. $\mathsf{CD}(K,N)$ is somehow related to the *fast diffusion equation* $\partial_t u = \Delta(u^{(N-1)/N})$, whereas it is always nonlinear. Moreover, even in the Riemannian setting, it is only partially known about the contraction property and gradient estimates corresponding to $\mathsf{CD}(K,N)$. An appropriate notion of $\mathsf{RCD}(K,N)$ should imply, for instance, the Li-Yau type gradient estimate.

References

[AGS1] L. Ambrosio, N. Gigli and G. Savaré, Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below, Preprint (2011). Available at arXiv:1106.2090

[AGS2] L. Ambrosio, N. Gigli and G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below, Preprint (2011). Available at arXiv:1109.0222

[BE] D. Bakry and M. Émery, Diffusions hypercontractives (French), Séminaire de probabilités, XIX, 1983/84, 177–206, Lecture Notes in Math. **1123**, Springer, Berlin, 1985.

[BCS] D. Bao, S.-S. Chern and Z. Shen, An introduction to Riemann-Finsler geometry, Springer-Verlag, New York, 2000.

[Ch] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. **9** (1999), 428–517.

[CC] J. Cheeger and T. H. Colding, On the structure of spaces with Ricci curvature bounded below. I, II, III, J. Differential Geom. **46** (1997), 406–480; ibid. **54** (2000), 13–35; ibid. **54** (2000), 37–74.

[CMS] D. Cordero-Erausquin, R. J. McCann and M. Schmuckenschläger, A Riemannian interpolation inequality à la Borell, Brascamp and Lieb, Invent. Math. **146** (2001), 219–257.

[Er] M. Erbar, The heat equation on manifolds as a gradient flow in the Wasserstein space, Ann. Inst. Henri Poincaré Probab. Stat. **46** (2010), 1–23.

[Fu] K. Fukaya, Collapsing of Riemannian manifolds and eigenvalues of Laplace operator, Invent. Math. **87** (1987), 517–547.

[Gi] N. Gigli, On the differential structure of metric measure spaces and applications, Preprint (2012). Available at arXiv:1205.6622

[GKO] N. Gigli, K. Kuwada and S. Ohta, Heat flow on Alexandrov spaces, to appear in Comm. Pure Appl. Math.

[Gr] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser, Boston, MA, 1999.

[JKO] R. Jordan, D. Kinderlehrer and F. Otto, The variational formulation of the Fokker-Planck equation, SIAM J. Math. Anal. **29** (1998), 1–17.

[Ku] K. Kuwada, Duality on gradient estimates and Wasserstein controls, J. Funct. Anal. **258** (2010), 3758–3774.

[Lo] J. Lott, Optimal transport and Ricci curvature for metric-measure spaces, Surveys in differential geometry **XI**, 229–257, Int. Press, Somerville, MA, 2007.

[LV1] J. Lott and C. Villani, Weak curvature conditions and functional inequalities, J. Funct. Anal. **245** (2007), 311–333.

[LV2] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. **169** (2009), 903–991.

[Mc1] R. J. McCann, A convexity principle for interacting gases, Adv. Math. **128** (1997), 153–179.

[Mc2] R. J. McCann, Polar factorization of maps on Riemannian manifolds, Geom. Funct. Anal. **11** (2001), 589–608.

[Oh1] S. Ohta, Gradient flows on Wasserstein spaces over compact Alexandrov spaces, Amer. J. Math. **131** (2009), 475–516.

[Oh2] S. Ohta, Finsler interpolation inequalities, Calc. Var. Partial Differential Equations **36** (2009), 211–249.

[Oh3] S. Ohta, Ricci curvature, entropy and optimal transport, to appear in Séminaires et Congrès. Available at arXiv:1009.3431

[Oh4] S. Ohta, Splitting theorems for Finsler manifolds of nonnegative Ricci curvature, Preprint (2012). Available at arXiv:1203.0079

[OS1] S. Ohta and K.-T. Sturm, Heat flow on Finsler manifolds, Comm. Pure Appl. Math. **62** (2009), 1386–1433.

[OS2] S. Ohta and K.-T. Sturm, Non-contraction of heat flow on Minkowski spaces, Arch. Ration. Mech. Anal. **204** (2012), 917–944.

[OS3] S. Ohta and K.-T. Sturm, Bochner-Weitzenböck formula and Li-Yau estimates on Finsler manifolds, Preprint (2011). Available at arXiv:1104.5276

[Ot] F. Otto, The geometry of dissipative evolution equations: the porous medium equation, Comm. Partial Differential Equations **26** (2001), 101–174.

[OV] F. Otto and C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality, J. Funct. Anal. **173** (2000), 361–400.

[vRS] M.-K. von Renesse and K.-T. Sturm, Transport inequalities, gradient estimates, entropy and Ricci curvature, Comm. Pure Appl. Math. **58** (2005), 923–940.

[Sa] G. Savaré, Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds, C. R. Math. Acad. Sci. Paris **345** (2007), 151–154.

[St1] K.-T. Sturm, On the geometry of metric measure spaces. I, Acta Math. **196** (2006), 65–131.

[St2] K.-T. Sturm, On the geometry of metric measure spaces. II, Acta Math. **196** (2006), 133–177.

[Vi1] C. Villani, Topics in optimal transportation, American Mathematical Society, Providence, RI, 2003.

[Vi2] C. Villani, Optimal transport, old and new, Springer-Verlag, 2009.